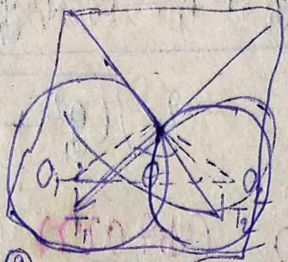
System of Circles

Chapter-1

Theorem :- 1

Qn. → To find the angle of intersection of two circles.

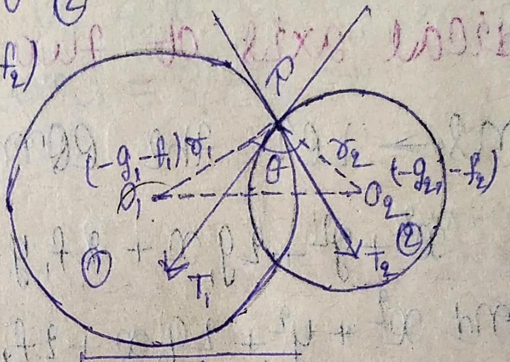
Ans.→ Let the eqn. of two circles be



$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad -(1)$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad -(2)$$

Let $O_1(-g_1,-f_1)$ and $O_2(-g_2,-f_2)$ be the centre of the given circle and their radius



$$r_1=\sqrt{g_1^2-f_1^2-c_1} \quad \text{and} \quad r_2=\sqrt{g_2^2+f_2^2-c_2}$$

Let P be the point of intersection of two circle from P draw PT_1 and PT_2 tangents to (1) and (2), we have,

Let $\angle T_1PT_2=\theta$

Joint O_1P, O_2P and O_1O_2

we know that the angle between two lines is equal of the angle between their normals.

$$\therefore \angle O_1PO_2=\theta, \text{ or, } \pi-\theta$$

$$O_1P=r_1=\sqrt{g_1^2+f_1^2-c_1}$$

$$O_2P=r_2=\sqrt{g_2^2+f_2^2-c_2}$$

$$O_1O_2 = \sqrt{(g_2-g_1)^2 + (f_2-f_1)^2}$$

Now. in ΔPO_1O_2

$$\cos\theta \text{ or, } \cos(\pi-\theta) = \frac{r_1^2 + r_2^2 - O_1O_2^2}{2r_1 \cdot r_2}$$

$$\cos\theta = \pm \frac{(g_1^2 + f_1^2 - C_1 + g_2^2 + f_2^2 - C_2 - (g_1-g_2)^2 + (f_1-f_2)^2}{2\sqrt{(g_1^2+f_1^2-C_1)}\sqrt{(g_2^2+f_2^2-C_2)}}$$

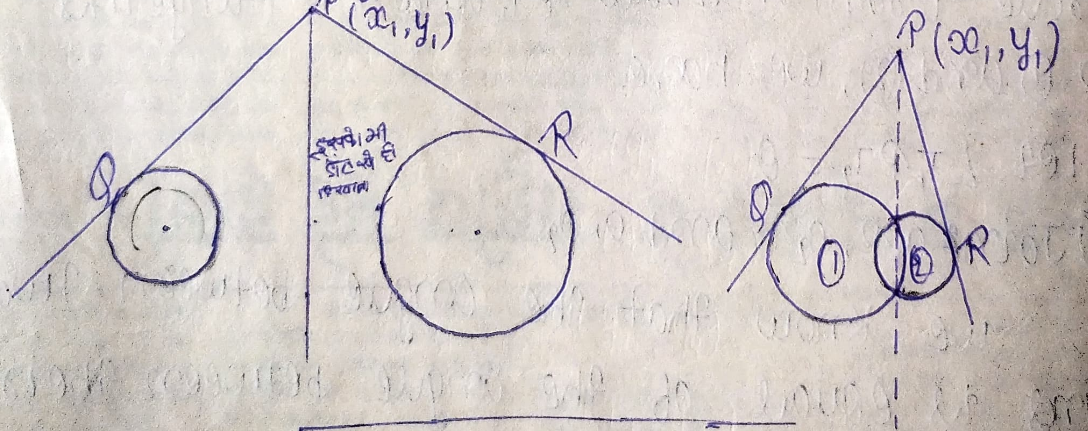
② QN.→ To obtain the equation of the radical axis of two circles.

Ans.→ Let the eqn. of two circle be

$$x^2+y^2+2g_1x+2f_1y+C_1=0 \quad -①$$

and $x^2+y^2+2g_2x+2f_2y+C_2=0 \quad -②$

Let $P(x_1, y_1)$ be any point from P drawn PQ and PR tangent to the circle ① and ②



$$\therefore PQ = \sqrt{x_1^2+y_1^2+2g_1x_1+2f_1y_1+C_1}$$

and $PR = \sqrt{x_1^2+y_1^2+2g_2x_1+2f_2y_1+C_2}$

from the defin of radical axis

$PQ = PR$

on squaring

$PQ^2 = PR^2$

$$\cancel{x_1^2} + \cancel{y_1^2} + 2g_1x_1 + 2f_1y_1 + c_1 = \cancel{x_1^2} + \cancel{y_1^2} + 2g_2x_1 + 2f_2y_1 + c_2$$

or, $2x_1(g_1 - g_2) + 2y_1(f_1 - f_2) + c_1 - c_2 = 0$

Hence, the locus of axis P i.e. radical axis is given.

$$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$

$S_1 = 0$ eqn. of 1st circle

$S_2 = 0$ eqn. of 2nd circle.

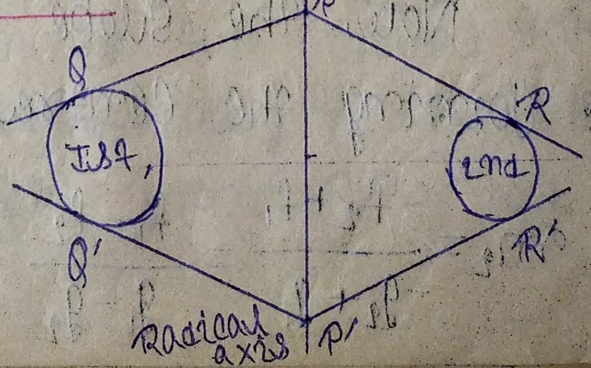
eqn. of radical axis $= S_1 - S_2 = 0$

(Regular 93) Theorem: 3

QN.→ Define radical axis of two circles and prove that radical axis of two circles is a line perpendicular to the line joining the centres of the two circles.

or, QN.→ Define radical axis. Prove that the radical axis of two circles is perpendicular to the line joining the centres of the circles.

Ans.→ Radical-axis:— Radical axis of two circles is the locus of points from which the tangents drawn



to the two circle are of equal length.

Let P be a point from P draw PQ and PR tangents to the Ist and 2nd circle which meets the circle at Q and R.

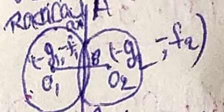
Hence, the locus of P is the radical axis of circle ① and ②.

Let the eqn. of two circle be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad — ①$$

and $$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad — ②$$

Let $O_1(-g_1,-f_1)$ and $O_2(-g_2,-f_2)$ be the centres of circle ① and ②



Hence the eqn. of radical axis of ① and ② is given by

$$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0$$

Now, m i.e. slope for gradiant of ③ radical axis, $m_1 = -\dfrac{2(g_1-g_2)}{2(f_1-f_2)}$

$$m_1 = -\frac{(g_1-g_2)}{(f_1-f_2)}$$

Now, the slope of O_1O_2 i.e. the line joining the centres of two circle

$$m_2 = \frac{-f_2+f_1}{-g_2+g_1} = \frac{f_1-f_2}{g_1-g_2}$$

$\because$ the slope of radical axis AB and the slope of O_1O_2, $m_1 m_2 = -\frac{(g_1-g_2)}{(f_1-f_2)} \times \frac{(f_1-f_2)}{(g_1-g_2)}$

$$m_1 \cdot m_2 = -1$$

Hence radical axis is $\perp$ to the line Joining their centres.

Theorem: 4

~~The Angle of intersection of two circles~~

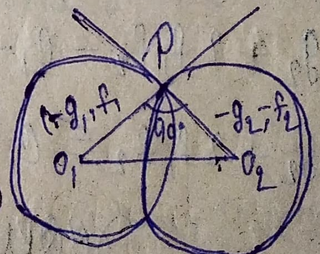
Qn.→ Define orthogonal circles. Find the necessary and sufficiet condition that the two circle, may cut one anoter ~~orthogon~~ orthogonally.

Ans.→ orthogonal-circles:- Two circles are said to cut orthogonally when the tangets at their common points of intersection include a right angle.

Let,

$x^2+y^2+2g_1x+2f_1y+c_1=0$ —(1)

and $x^2+y^2+2g_2x+2f_2y+c_2=0$ —(2)



be the eqn. of two circles intersection each other at P orthogonally i.e the angle betwen the tangent from P to both circle = 90°

Hence, these tangent must passing through the centre O_1 and O_2 of the circle

the Co-ordinates of centre O_1 and O_2 we have

$(-g_1, -f_1)$ and $(-g_2, -f_2)$ and

radius $O_1P = \sqrt{g_1^2 + f_1^2 - c_1}$ and $O_2P = \sqrt{g_2^2 + f_2^2 - c_2}$

Now, $O_1P^2 = g_1^2 + f_1^2 - c_1$

$O_2P^2 = g_2^2 + f_2^2 - c_2$

$O_1O_2^2 = (g_2 - g_1)^2 + (f_2 - f_1)^2$

As the two circle cut orthogonally at P $\therefore \angle O_1PO_2 = 90^\circ$

$\therefore O_1O_2^2 = O_1P^2 + O_2P^2$

or, $(g_2 - g_1)^2 + (f_2 - f_1)^2 = g_1^2 + f_1^2 + c_1 + g_2^2 + f_2^2 - c_2$

or, $\cancel{g_2^2} + \cancel{g_1^2} - 2g_1g_2 + \cancel{f_2^2} + \cancel{f_1^2} - 2f_1f_2 = \cancel{g_1^2} + \cancel{f_1^2} - c_1 + \cancel{g_2^2} + \cancel{f_2^2} - c_2$

$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0$

or, $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

required condition this is necessary condition.

Theorem: 5

To find the general eqn. of the circles cutting two given circles orthogonally.

Ans.→ Let the eqn. of two given circle be $x^2+y^2+2g_1x+2f_1y+c_1=0$ — (1)

and $x^2+y^2+2g_2x+2f_2y+c_2=0$ — (2)

Let $x^2+y^2+2gx+2fy+c=0$

or, $2gx+2fy+x^2+y^2+c$ — (3), be the eqn. of that circle which cuts both (1) and (2) orthogonally.

As, (1) and (3) are orthognal

$2gg_1+2ff_1=c_1+c$

or, $2gg_1+2ff_1-c_1-c=0$ — (4)

As (2) and (3) are orthognal

$\therefore 2gg_2+2ff_2-c_2-c=0$ — (5)

Now, elliminatary g and f from (3), (4) and (5)

$$\begin{vmatrix} x^2+y^2+c \\ -c_1-c \\ -c_2-c \end{vmatrix}=0$$

required condition